HADRONIC WEAK INTERACTIONS OF LIGHT QUARKS

G. Martinelli [a]

[a]Dip. di Fisica, Univ. "La Sapienza" and INFN, P.le A. Moro, I-00185 Rome, Italy

In this review, three subjects are discussed: a phenomenological application of lattice predictions to K^0–$\bar{K}^0$ mixing in Super Symmetry; a discussion of the non-perturbative renormalization methods for four-fermion operators and a new approach to extract weak matrix elements in effective theories denoted as OPE without OPE (operator product expansion without operator product expansion).

1. Introduction

Since the original proposals of using lattice QCD to study hadronic weak decays [1]–[3], substantial theoretical and experimental progress has been made: with both Wilson and staggered Fermions, the main theoretical aspects of the renormalization of composite four-fermion operators are, by now, fully understood [4,5]; the calculation of the K^0–$\bar{K}^0$ mixing amplitude, expressed in terms of the so-called renormalization group invariant B-parameter $\hat{B}_K$, has reached a level of precision (at least in the quenched approximation) which is unpaired by any other approach [6]–[8] (for a detailed discussion see [9]); increasing precision has also been gained in the determination of the electro-weak penguin amplitudes (and of the strange quark mass [10]), which are relevant in the prediction of the CP violating parameter ϵ'/ϵ. A lot of progress has also been made in the determination of quantities which enter heavy hadron weak processes and which are reviewed by T. Draper in his plenary talk [11]. Still, in particular for kaon physics, many problems are unsolved or poorly understood. By using chiral perturbation theory and $SU(3)$ symmetry, for example, one may relate $\hat{B}_K$ to the physical $\Delta I = 3/2$ $K \to \pi\pi$ amplitude, A_2. The large value of $\hat{B}_K \sim 0.85$ found in lattice calculations leads to a prediction for A_2 which is about 40% larger than its experimental value. It is not clear whether this is due to a failure of chiral perturbation theory, or to electro-magnetic effects [12]. In the former case, the use of $K \to \pi$ matrix elements, combined with chiral perturbation theory, to predict the $\Delta I = 1/2$ amplitude A_0 or ϵ'/ϵ would be very suspicious. Until very recently, the only existing calculation of two other B-parameters which are very important for ϵ'/ϵ, namely B_5 and B_6, was the one performed with staggered Fermions in [13]. Then, the perturbative one-loop lattice corrections to the relevant operators were computed [14] and found so large as to make the old estimate of [13] very doubtful (a recent calculation of B_6 was presented in [15]). Thus, we do not have any reliable lattice estimate of B_5 and B_6. Finally, there is no convincing lattice calculation of A_0 yet, although some new results with staggered fermions have been presented at this Conference [16].

In spite of considerable technical improvements, and of the increasing computer power, no fundamental advance has been done last years for the problems mentioned above. For this reason, rather than presenting new results, I discuss new methods which have been proposed in order to compute the relevant matrix elements. In most of the cases, these methods have not been applied yet, or only feasibility studies exist. This is the content of sec. 4. In sec. 2, I present a recent application of lattice calculations of $\Delta S = 2$ amplitudes to flavor changing neutral current (FCNC) phenomenology in Super Symmetric models. In sec. 3 several approaches, which have been used or proposed for the non-perturbative renormalization of lattice four-fermion operators, are discussed and compared.

2. $\Delta S = 2$ transitions in SUSY

The prescription of minimality in the number of new particles introduced to supersymmetrise the Standard Model (SM), together with the demand of conservation of baryon and lepton numbers, does not prevent the appearance of more than 100 new SUSY parameters, in addition to the 18 already present in the SM. FCNC and CP violating phenomena are protagonists of a drastic reduction of these extra degrees of freedom. Among the possible FCNC processes, K^0–$\bar{K}^0$ (B^0–$\bar{B}^0$) mixing play a very special role. In this section, a recent analysis of K^0–$\bar{K}^0$ mixing in SUSY is discussed [17]. This analysis makes use of the recently computed NLO corrections to the SUSY effective Hamiltonian [18] and of the matrix elements of the relevant operators (renormalized non-perturbatively in the RI-MOM scheme), computed on the lattice [19]. This is a new, interesting application of the lattice method for studying the physics beyond the SM. Unfortunately, for lack of space, many other applications to physics beyond the SM cannot be discussed here. A detailed discussion of the non-perturbative methods which can be used to renormalize the relevant four-fermion operators can be found in sec. 3.

Let us start by defining the model used to construct the low-energy effective Hamiltonian for $\Delta F = 2$ transitions. In the so-called mass insertion approximation [20], one chooses the super-CKM basis for the fermion and sfermion states, where all the couplings of these particles to neutral gauginos are flavour diagonal, while the genuine SUSY FC effects are exhibited by the non-diagonality of the sfermion mass matrices. Denoting by Δ^2 the off-diagonal terms in the sfermion mass matrices (i.e. the mass terms relating sfermions of the same electric charge, but different flavour), the sfermion propagators can be expanded as a series in terms of $\delta = \Delta^2/\tilde{m}^2$, where $\tilde{m}$ is the average sfermion mass. As long as Δ^2 is significantly smaller than $\tilde{m}^2$, we can just take the first term of this expansion and, then, the experimental information concerning FCNC and CP violating phenomena translates into upper bounds on the δs [21]–[22]. There exist four different Δ mass-insertions connecting flavours d and s

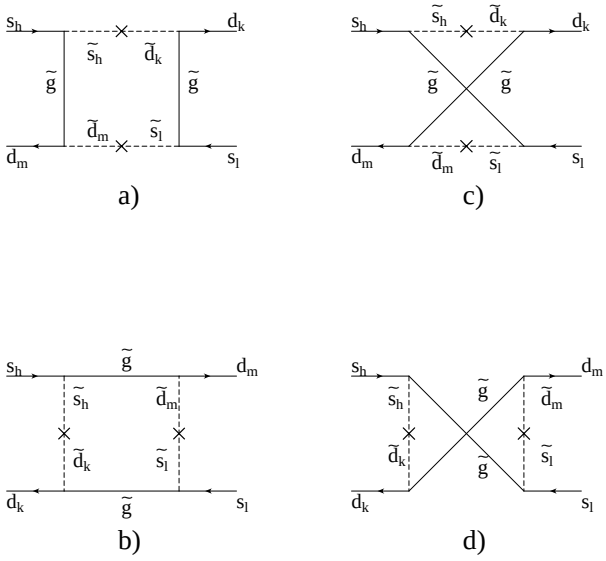

Figure 1. Feynman diagrams for $\Delta S = 2$ transitions, with $h, k, l, m = \{L, R\}$.

along a sfermion propagator: $\left(\Delta^d_{12}\right)_{LL}$, $\left(\Delta^d_{12}\right)_{RR}$, $\left(\Delta^d_{12}\right)_{LR}$ and $\left(\Delta^d_{12}\right)_{RL}$. The indices L and R refer to the helicity of the fermion partners. For gluino-mediated processes, for example, the amplitude for $\Delta S = 2$ transitions in the full theory is given by the computation of the diagrams in fig. 1. Having calculated the amplitude from the diagrams in fig. 1, one has to choose a basis of local operators and perform the matching of the full theory to the one described by $\mathcal{H}_{\rm eff}^{\Delta S=2}$. In [17,19], they have adopted the form

$$\mathcal{H}_{\rm eff}^{\Delta S=2} = \sum_{i=1}^{5} C_i\, Q_i + \sum_{i=1}^{3} \tilde{C}_i\, \tilde{Q}_i \qquad (1)$$

where

$$\begin{aligned} Q_1 &= \bar{d}^\alpha_L \gamma_\mu s^\alpha_L \bar{d}^\beta_L \gamma^\mu s^\beta_L\,, \quad Q_2 = \bar{d}^\alpha_R s^\alpha_L \bar{d}^\beta_R s^\beta_L\,, \\ Q_3 &= \bar{d}^\alpha_R s^\beta_L \bar{d}^\beta_R s^\alpha_L\,, \\ Q_4 &= \bar{d}^\alpha_R s^\alpha_L \bar{d}^\beta_L s^\beta_R\,, \quad Q_5 = \bar{d}^\alpha_R s^\beta_L \bar{d}^\beta_L s^\alpha_R\,, \end{aligned} \qquad (2)$$

and the operators $\tilde{Q}_{1,2,3}$ are obtained from the $Q_{1,2,3}$ by the exchange $L \leftrightarrow R$. Here $q_{R,L} =$

$P_{R,L}\,q$, with $P_{R,L} = (1 \pm \gamma_5)/2$, and α and β are colour indices. The Wilson coefficients $C_i(\mu)$ ($\tilde{C}_i(\mu)$) for the operators renormalized at the scale μ are obtained by computing the coefficients at the large energy scale, $C_i(M_S)$, where M_S is the order of the gluino (squark) mass, and then evolving them to μ using suitable QCD renormalization group equations, which depend on the operator anomalous-dimension matrix. The latter is only determined from QCD and does not know about the underlying fundamental theory at the large energy scale. The next-to-leading anomalous-dimension matrix for the most general $\mathcal{H}_{\rm eff}^{\Delta F=2}$ has been computed in [18]. They used the regularisation-independent (RI) anomalous dimension in the Landau gauge, since we will make use of matrix elements computed in lattice QCD with the same choice of renormalisation scheme [19]. A full NLO computation would also require the $O(\alpha_s)$ corrections to the matching conditions. Unfortunately, such corrections are not available yet. One might argue that, being of order $\alpha_s(M_S)$, these contributions should be small, as suggested by the cases of the SM and of the two Higgs doublet model; this statement can only be confirmed by an explicit computation. Due to the absence of $O(\alpha_s)$ corrections to the matching, our $\mathcal{H}_{\rm eff}^{\Delta F=2}$ will be affected by a residual scheme dependence, which would be cancelled by the missing terms of order $\alpha_s(M_S)$. A simple analytic formula for the expression of the Wilson coefficients as a function of the initial conditions at the SUSY scale, $C(M_S)$, and of $\alpha_s(M_S)$ can be given. For $M_S > m_t$, one has

$$C_r(\mu) = \sum_i \sum_s \left(b_i^{(r,s)} + \eta\, c_i^{(r,s)} \right) \eta^{a_i}\, C_s(M_S) \quad (3)$$

where, in the evolution of the coefficients from M_S, $M_S = (M_{\tilde{g}} + M_{\tilde{q}})/2$ has been chosen. $\eta = \alpha_s(M_S)/\alpha_s(m_t)$, and the magic numbers a_i, $b_i^{r,s}$ and $c_i^{r,s}$ at $\mu = 2$ GeV in the RI-MOM scheme can be found in [17]. The magic numbers for the evolution of $\tilde{C}_{1-3}$ are the same as the ones for the evolution of C_{1-3}. Eq. (3) can be used in connection with the B-parameters, given in eq. (5) below, to compute the contribution to ΔM_K and ε_K at the NLO for any model of new physics in which the new contributions with respect to the SM originate from extra heavy particles. One has just to plug in (5) the C_i of his favourite model. In the case of the renormalised operators, a convenient definition of the B-parameters is the following

$$\begin{aligned}
\frac{\langle \bar{K}^0|\hat{Q}_1(\mu)|K^0\rangle}{M_K f_K^2} &= \frac{B_1}{3} \\
\frac{\langle \bar{K}^0|\hat{Q}_2(\mu)|K^0\rangle}{M_K f_K^2} &= -\left(\frac{M_K}{m_s(\mu)+m_d(\mu)}\right)^2 \frac{5B_2}{24} \\
\frac{\langle \bar{K}^0|\hat{Q}_3(\mu)|K^0\rangle}{M_K f_K^2} &= \left(\frac{M_K}{m_s(\mu)+m_d(\mu)}\right)^2 \frac{B_3}{24} \\
\frac{\langle \bar{K}^0|\hat{Q}_4(\mu)|K^0\rangle}{M_K f_K^2} &= \left(\frac{M_K}{m_s(\mu)+m_d(\mu)}\right)^2 \frac{B_4}{4} \\
\frac{\langle \bar{K}^0|\hat{Q}_5(\mu)|K^0\rangle}{M_K f_K^2} &= \left(\frac{M_K}{m_s(\mu)+m_d(\mu)}\right)^2 \frac{B_5}{12}
\end{aligned} \quad (4)$$

where the notation $\hat{Q}_i(\mu)$ (or simply $\hat{Q}_i$) denotes the operators renormalised at the scale μ. Here and in the following, the same expressions of the B-parameters of the operators Q_{1-3} are valid for the operators $\tilde{Q}_{1-3}$. In eq. (4) the operators and the quark masses are renormalised in the same scheme (RI, $\overline{MS}$, etc.) at the scale μ and the numerical results for the B-parameters, $B_i = B_i(\mu)$, presented below refer to the Landau RI scheme. Moreover, without loss of generality, terms which are of higher order in the chiral expansion and which are usually included in the definition of the B-parameters have been omitted. The advantage with the definition (4) is that the B-parameters obeys to very simple renormalization group equations, with an anomalous dimension matrix which is related in a simple way to that of the corresponding operators [19]. In the numerical study below, the following values of the B-parameters, for $\mu = 2$ GeV, have been used

$$\begin{aligned}
B_1 &= 0.60(6) \quad B_2 = 0.66(4) \\
B_3 &= 1.05(12) \\
B_4 &= 1.03(6) \quad B_5 = 0.73(10)
\end{aligned} \quad (5)$$

The central value used for $B_1 = B_K$ corresponds to $B_K^{\overline{MS}}(2\,\mathrm{GeV}) = 0.61$, in agreement with the recent estimates of [9]. B_{2-5} have been taken from [19], where all details of the computation can

be found (for another determination of these B-parameters, calculated with a perturbative renormalization, see [23]).
As an example, I present the results of a model-independent analysis of K^0–$\bar{K}^0$ mixing. The K_L–K_S mass difference ΔM_K and the CP-violating parameter ε_K are given by

$$\Delta M_K = 2Re\ \langle K^0|\mathcal{H}_{\rm eff}^{\Delta S=2}|\bar{K}^0\rangle\ , \qquad (6)$$

$$\varepsilon_K = \frac{1}{\sqrt{2}\Delta M_K} Im\ \langle K^0|\mathcal{H}_{\rm eff}^{\Delta S=2}|\bar{K}^0\rangle\ . \qquad (7)$$

The SUSY (gluino-mediated) contribution to the low-energy $\mathcal{H}_{\rm eff}^{\Delta S=2}$ contains two real and four complex unknown parameters: $m_{\tilde{q}}$, $m_{\tilde{g}}$, $\left(\delta^d_{12}\right)_{LL}$, $\left(\delta^d_{12}\right)_{LR}$, $\left(\delta^d_{12}\right)_{RL}$ and $\left(\delta^d_{12}\right)_{RR}$. The model-independent constraints are obtained by imposing that the sum of the SUSY contributions proportional to a single δ-parameter and of the SM contributions to ΔM_K and ε_K does not exceed the experimental value for these quantities. This is justified by noting that the constraints on different δ-parameters in the kaon case exhibit a hierarchical structure, and therefore interference effects between different contributions would require a large amount of fine tuning. The B-parameters used in the analysis are those given in eq. (5) subtracted by one standard deviation to their central values, in order to extract a more conservative bound on SUSY parameters. The limits on the δ-parameters for a typical value of the gluino and squark masses are reported in tables 1 and 2 (more detail can be found in [17]).
As a glimpse at tables 1 and 2 readily shows, the inclusion of the NLO QCD corrections to the Wilson coefficients of $\mathcal{H}_{\rm eff}^{\Delta S=2}$ and the use of the lattice B-parameters, instead than the matrix elements evaluated in the vacuum saturation approximation (VSA), affects the results in different ways, according to the different operators. The effects are particularly large for left-right operators.

3. Non-perturbative renormalization of four-fermion operators

From the theoretical point of view, the calculation of hadronic weak decays can usually be divided in two parts: the calculation of the effects of

	NO QCD, VSA	NLO, Lattice B_i
x	$\sqrt{\lvert Re(\delta^d_{12})^2_{LL}\rvert}$	
0.3	1.4×10^{-2}	2.2×10^{-2}
1.0	3.0×10^{-2}	4.6×10^{-2}
4.0	7.0×10^{-2}	1.1×10^{-1}
x	$\sqrt{\lvert Re(\delta^d_{12})^2_{LR}\rvert}$	$(\lvert(\delta^d_{12})_{LR}\rvert \gg \lvert(\delta^d_{12})_{RL}\rvert)$
0.3	3.1×10^{-3}	2.6×10^{-3}
1.0	3.4×10^{-3}	2.8×10^{-3}
4.0	4.9×10^{-3}	3.9×10^{-3}
x	$\sqrt{\lvert Re(\delta^d_{12})^2_{LR}\rvert}$	$((\delta^d_{12})_{LR} = (\delta^d_{12})_{RL})$
0.3	5.5×10^{-3}	1.7×10^{-3}
1.0	3.1×10^{-3}	2.8×10^{-2}
4.0	3.7×10^{-3}	3.5×10^{-3}
x	$\sqrt{\lvert Re(\delta^d_{12})_{LL}(\delta^d_{12})_{RR}\rvert}$	
0.3	1.8×10^{-3}	8.6×10^{-4}
1.0	2.0×10^{-3}	9.6×10^{-4}
4.0	2.8×10^{-3}	1.3×10^{-3}

Table 1
Limits on $\mathrm{Re}\left(\delta_{ij}\right)_{AB}\left(\delta_{ij}\right)_{CD}$, with $A, B, C, D = (L, R)$, for an average squark mass $m_{\tilde{q}} = 500$ GeV and for different values of $x = m^2_{\tilde{g}}/m^2_{\tilde{q}}$.

	NO QCD, VSA	NLO, Lattice B_i
x	$\sqrt{\lvert Im(\delta^d_{12})^2_{LL}\rvert}$	
0.3	1.8×10^{-3}	2.9×10^{-3}
1.0	3.9×10^{-3}	6.1×10^{-3}
4.0	9.2×10^{-3}	1.4×10^{-2}
x	$\sqrt{\lvert Im(\delta^d_{12})^2_{LR}\rvert}$	$(\lvert(\delta^d_{12})_{LR}\rvert \gg \lvert(\delta^d_{12})_{RL}\rvert)$
0.3	4.1×10^{-4}	3.4×10^{-4}
1.0	4.6×10^{-4}	3.7×10^{-4}
4.0	6.5×10^{-4}	5.2×10^{-4}
x	$\sqrt{\lvert Im(\delta^d_{12})^2_{LR}\rvert}$	$((\delta^d_{12})_{LR} = (\delta^d_{12})_{RL})$
0.3	7.2×10^{-4}	2.2×10^{-4}
1.0	4.1×10^{-4}	3.7×10^{-3}
4.0	4.9×10^{-4}	4.7×10^{-4}
x	$\sqrt{\lvert Im(\delta^d_{12})_{LL}(\delta^d_{12})_{RR}\rvert}$	
0.3	2.3×10^{-4}	1.1×10^{-4}
1.0	2.6×10^{-4}	1.3×10^{-4}
4.0	3.7×10^{-4}	1.8×10^{-4}

Table 2
Limits on $\mathrm{Im}\left(\delta_{ij}\right)_{AB}\left(\delta_{ij}\right)_{CD}$, with $A, B, C, D = (L, R)$, for an average squark mass $m_{\tilde{q}} = 500$ GeV and for different values of $x = m^2_{\tilde{g}}/m^2_{\tilde{q}}$.

strong interactions at short distances, which can be done in perturbation theory using the Wilson Operator Product Expansion (OPE), and the calculation of the hadronic matrix elements of local operators. The latter contain the effects of strong forces in the infrared region where perturbation theory cannot be applied and the lattice approach is used. This is the path followed in sec. 2 for the costruction of $\mathcal{H}_{\rm eff}^{\Delta F=2}$. An important point in the full approach is the consistency between the procedure used to obtain the renormalized operators and the scheme adopted in the calculation of the Wilson coefficients. Unlike in many other formulations, as for example in the $1/N$ expansion beyond the leading order [24], and in the chiral quark model [25], the matching of the renormalized operators to the corresponding Wilson coefficients can be rigorously done, at least in principle, using lattice perturbation theory. In practice, the matrix elements of the renormalized operators constructed from the bare lattice ones are subject to two main sources of systematic effects: the renormalization constants (matrices) are usually computed in one-loop lattice perturbation theory and are subject to large $\mathcal{O}(g^4)$ errors; in most of the cases the matrix elements are computed using the Wilson action and they suffer from $\mathcal{O}(a)$ discretization errors.

Numerical studies [27] of the matrix element of the $\Delta S = 2$ left-left operator, $O^{\Delta S=2} = \bar{s}\gamma_\mu(1-\gamma_5)d\,\bar{s}\gamma_\mu(1-\gamma_5)d$, have shown that the chiral behaviour of the renormalized operator is not significantly better if one adopt a tree-level improved action and operators, whereas it is significantly improved by using non-perturbatively determined renormalization constants (either by Ward identities [26] or with the non-perturbative method (NPM) of [28]) [1].

In this section, I review some of the methods which is possible to use in order to renormalize non-perturbatively the lattice operators, thus eliminating one of the main sources of uncertainties in the calculation of the physical amplitudes. To this purpose it is useful to start with some definition and classification of the relevant four fermion operators. In the following I only discuss the case of $\Delta S = 2$ or $\Delta I = 3/2$ operators, which cannot mix with operators of lower dimensions. A full discussion of the renormalization of $\Delta I = 1/2$ operators can be found in [29] and more recently in [30], see also [36].

Operator basis In order to illustrate the renormalization properties of these operators, it is convenient to work in a theory with four distinct flavours and define the generic operator as

$$O^{\pm}_{\Gamma_1\Gamma_2} \equiv \frac{1}{2}\left[O_{\Gamma_1\Gamma_2} \pm O^F_{\Gamma_1\Gamma_2}\right] = \frac{1}{2}\left[(\bar{\psi}_1\Gamma_1\psi_2)(\bar{\psi}_3\Gamma_2\psi_4) \pm (\bar{\psi}_1\Gamma_1\psi_4)(\bar{\psi}_3\Gamma_2\psi_2)\right] \tag{8}$$

where $\Gamma_{1,2}$ denote one of the 16 Dirac matrices, $\Gamma = \{\mathbf{I}, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5\} \equiv \{S, V, T, A, P\}$, and summation of Lorentz indices is implied when necessary. For example O_{VV} corresponds to $VV \equiv \sum_\mu \gamma_\mu \otimes \gamma_\mu$, $VA \equiv \sum_\mu \gamma_\mu \otimes \gamma_\mu\gamma_5$, etc. In total we can classify ten parity-even and ten parity-odd operators. There are several possible choices for the operator basis. In the following, I will use

$$\begin{aligned} Q_1^\pm &\equiv O^\pm_{[VV+AA]} \quad Q_2^\pm \equiv O^\pm_{[VV-AA]} \\ Q_3^\pm &\equiv O^\pm_{[SS-PP]} \quad Q_4^\pm \equiv O^\pm_{[SS+PP]} \\ Q_5^\pm &\equiv O^\pm_{TT} \end{aligned} \tag{9}$$

for the parity-even operators and

$$\begin{aligned} \mathcal{Q}_1^\pm &\equiv -O^\pm_{[VA+AV]} \quad \mathcal{Q}_2^\pm \equiv O^\pm_{[VA-AV]} \\ \mathcal{Q}_3^\pm &\equiv -O^\pm_{[SP-PS]} \quad \mathcal{Q}_4^\pm \equiv O^\pm_{[SP+PS]} \\ \mathcal{Q}_5^\pm &\equiv O^\pm_{T\tilde{T}} \end{aligned} \tag{10}$$

for the parity-odd ones.

Parity, chirality and generalized CPS symmetries restrict the possible mixing between different operators [31]. The simplest case is that of the parity-odd operators for which the mixing matrix, $\mathcal{Z}_{ij}$, is a sparse block diagonal one. We have

$$\hat{\mathcal{Q}}^\pm = \mathcal{Z}^\pm\mathcal{Q}^\pm \tag{11}$$

where the $\hat{\mathcal{Q}}_i$ denote the renormalized operators

[1] For a different conclusion, using a boosted improved action and operators, see L. Lellouch at this Conference.

and

$$\mathcal{Z}^{\pm} = \begin{pmatrix} \mathcal{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22} & \mathcal{Z}_{23} & 0 & 0 \\ 0 & \mathcal{Z}_{32} & \mathcal{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ 0 & 0 & 0 & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}^{\pm} \tag{12}$$

The block-structure of the matrix $\mathcal{Z}_{ij}$ is not changed by a lattice regularization which breaks chiral symmetry. The situation is rather different for the parity-even operators Q_i. In an ideal, chirally symmetric regularization scheme, χRS, the mixing matrix of the parity-even operators Z_{ij} would have the same block-structure as the mixing matrix of the parity-odd ones. The explicit chiral symmetry breaking induces further mixing which can be removed by suitable counterterms which enforce, up to discretization errors, the relevant Ward identities for the subtracted operators. A convenient parametrization of the renormalization matrix is given by the following expressions

$$\hat{Q}^{\pm} = Z^{\pm} Q^{\pm} = Z_{\chi}^{\pm} \tilde{Q}^{\pm} \tag{13}$$

$$\tilde{Q}^{\pm} = [I + \Delta^{\pm}] Q^{\pm} \tag{14}$$

where

$$Z_{\chi}^{\pm} = \begin{pmatrix} Z_{11} & 0 & 0 & 0 & 0 \\ 0 & Z_{22} & Z_{23} & 0 & 0 \\ 0 & Z_{32} & Z_{33} & 0 & 0 \\ 0 & 0 & 0 & Z_{44} & Z_{45} \\ 0 & 0 & 0 & Z_{54} & Z_{55} \end{pmatrix}^{\pm} \tag{15}$$

and

$$\Delta^{\pm} = \begin{pmatrix} 0 & \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{15} \\ \Delta_{21} & 0 & 0 & \Delta_{24} & \Delta_{25} \\ \Delta_{31} & 0 & 0 & \Delta_{34} & \Delta_{35} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & 0 & 0 \\ \Delta_{51} & \Delta_{52} & \Delta_{53} & 0 & 0 \end{pmatrix}^{\pm} \tag{16}$$

Note that in the hypothetical χRS, $\Delta^{\pm} = 0$ and $\mathcal{Z}_{ij} = Z_{ij}$.

Ward identities and the NPM I now discuss, with one simple example, the determination of the mixing matrix $\Delta^{\pm}$ using the Ward identity method (WIM). The WIM, first discussed in [4] (see also [32]), was applied in [33] to determine the renormalization constant of the axial current using quark Green functions and then implemented to the non-perturbative determination of the mixing coefficients of the $\Delta S = 2$ operator Q_1 in [26]. I will also show that the WIM is equivalent to the NPM on quark and gluon states, first proposed in [28] and then implemented for the renormalization of the four-fermion operators in [27]. A more detailed discussion can be found in [34]. Before discussing the Ward identities, I have to introduce some definition. The Green functions considered in the following are given by the expectation values of the following multilocal operators:

$$\begin{aligned} G_k(x_0; x_{1,2,3,4}) &= \psi_1 \bar{\psi}_2 Q_k(x_0) \psi_3 \bar{\psi}_4 \\ \mathcal{G}_k(x_0; x_{1,2,3,4}) &= \psi_1 \bar{\psi}_2 \mathcal{Q}_k(x_0) \psi_3 \bar{\psi}_4 \\ \mathcal{G}_k^{(1)}(x_0; x_{1,2,3,4}) &= [\gamma_5 \psi_1] \bar{\psi}_2 \mathcal{Q}_k(x_0) \psi_3 \bar{\psi}_4 \\ \mathcal{G}_k^{(2)}(x_0; x_{1,2,3,4}) &= \psi_1 [\bar{\psi}_2 \gamma_5] \mathcal{Q}_k(x_0) \psi_3 \bar{\psi}_4 \end{aligned}$$

and similarly for $\mathcal{G}_k^{(3)}$ and $\mathcal{G}_k^{(4)}$; $k = 1, \ldots, 5$. In the above equation $\psi_1 = \psi_1(x_1)$, $\psi_2 = \psi_2(x_2)$, etc. For simplicity, the colour and spin indices have not been shown explicitly and the $\pm$ superscripts have been dropped from operators and correlation functions. Having thus defined G_k, $\mathcal{G}_k$ $\mathcal{G}_k^{(1)}$ etc. in coordinate space, we will also be using the corresponding Green functions in momentum space $G_k(p)$, $\mathcal{G}_k(p)$, $\mathcal{G}_k^{(1)}(p)$ etc. (all external legs having the same Euclidean momentum). The amputated Green functions of $\langle G_k(p) \rangle$, $\langle \mathcal{G}_k(p) \rangle$ and $\langle \mathcal{G}_k^{(1)}(p) \rangle$ will be denoted by $\Lambda_k(p)$, $\mathcal{J}\!\!\!\backslash_k(p)$ and $\mathcal{J}\!\!\!\backslash_k^{(1)}(p)$, respectively.

On quark states, the relevant Ward identity for the operator Q_1 can be written as

$$\begin{aligned} \langle G_1 \rangle - \frac{1}{4} \left(\langle \mathcal{G}_1^{(1)} \rangle - \langle \mathcal{G}_1^{(2)} \rangle + \langle \mathcal{G}_1^{(3)} \rangle - \langle \mathcal{G}_1^{(4)} \rangle \right) = \\ - \int d^4x \langle \mathcal{G}_1 \left[\nabla^{\mu} A_{\mu}(x) - 2 m_0 P(x) - X_5(x) \right] \rangle \end{aligned} \tag{17}$$

where the flavour indices of $A_{\mu}(x)$ and $P(x)$ have been omitted. Out of the chiral limit, the $2m_0 P(x)$ term is present, but the surface term from the current divergence vanishes upon integration. I will be considering this case, in order to mimic what is practicaly done in numerical simulations (i.e. first we compute quantities

at small non-zero quark mass and then extrapolate to the chiral limit). The operator X_5 arises from the variation of the chiral symmetry breaking Wilson term in the action. As shown in [4] (see also [35] for a detailed discussion) it mixes, under renormalization with $\nabla^\mu A_\mu$ and P. This mixing, determined by the requirement that on-shell matrix elements of the subtracted X_5 vanish in the continuum, generates a finite renormalization of the axial current and a power subtraction of the quark mass. Thus, following [4], in the above WI we will trade-off X_5 for the renormalized expression $[\nabla^\mu \hat{A}_\mu - 2\hat{m}\hat{P} - \overline{X}_5]$, where $\overline{X}_5$ is the subtracted X_5. What is of interest to us is that, besides the above renormalizations, the $\overline{X}_5$ insertion in the above corelation function also generates new contact terms

$$\begin{aligned}
&\langle \overline{X}_5(x)\mathcal{G}_1(x_0;x_1,x_2,x_3,x_4)\rangle = \\
&(1-\frac{Z_{11}}{\mathcal{Z}_{11}})\langle G_1\rangle\delta(x-x_0) \\
&-\frac{Z_{11}}{\mathcal{Z}_{11}}\sum_{k=2}^{5}\Delta_{1k}\langle G_k\rangle\delta(x-x_0) \qquad (18)
\end{aligned}$$

where the notation for the various coefficients has been chosen with some foresight. There are also contact terms arising from the proximity of $\overline{X}_5(x)$ to the quark fields of the correlation $\mathcal{G}_1$ at points $x_1,\cdots,x_4$. These terms have the form $\langle \mathcal{G}_1^{(k)}\rangle\delta(x-x_k)$ (with $k=1,\cdots,4$). However, as shown in [36], they vanish in the continuum limit. We now combine eqs. (17) and (18), Fourier-transform the WI (with all external momenta set equal to p) and amputate the resulting correlation functions

$$\begin{aligned}
&\tilde{\Lambda}_1(p) = \Lambda_1(p) + \sum_{k=2}^{5}\Delta_{1k}\Lambda_{1k}(p) = \\
&\frac{\mathcal{Z}_{11}}{4Z_{11}}[\mathcal{\Lambda}_1^{(1)} - \mathcal{\Lambda}_1^{(2)} + \mathcal{\Lambda}_1^{(3)} - \mathcal{\Lambda}_1^{(4)}] \\
&+\frac{\mathcal{Z}_{11}}{Z_{11}}\int d^4x\langle \mathcal{G}_1(p)\hat{m}\hat{P}(x)\rangle\prod_{j=1}^{4}\langle S^{-1}(p)\rangle \qquad (19)
\end{aligned}$$

We require the above WI, up to quark field renormalization, to be the one valid in the continuum limit for subtracted correlation functions (operators). This implies that the l.h.s., multiplied by Z_{11}, can be identified with the renormalized parity-conserving correlation function (operator). We see immediately that the WI fixes the operator subtractions (i.e. the mixing coefficients Δ_{1k}). This can be done by projecting the above WI with suitable projectors [34] and by solving the resulting system of linear non-homogenous equations for the four Δ's [2]. The WI also fixes the ratio of the multiplicative renormalization constants $Z_{11}/\mathcal{Z}_{11}$, but not each of them separately. This is the analog of what happens in the case of the scalar and pseudoscalar densities: the Ward identities can only fix the ratio of the renormalization constants of these operators, Z_P/Z_S, but not Z_P or Z_S. Their values, for fixed Z_P/Z_S, is arbitrary and depend on the renormalization scheme.

The WI (19) is only a function of bare quantities. This demonstrates that the matrix Δ and the ratio $Z_{11}/\mathcal{Z}_{11}$ are finite functions of g_0^2, which can be completely determined from the WIs, and do not depend on the renormalization conditions. The overall renormalization constant Z_{11} ($\mathcal{Z}_{11}$), instead, is logarithmically divergent (as in the continuum) and is fixed by imposing a suitable (scheme-dependent) renormalization condition on the subtracted operator $\tilde{Q}_1$.

I will demonstrate now that the WIM and the NPM are equivalent for the determination of the lattice mixing coefficients. The validity of this statement has already been discussed in general in [28] and in the specific case of the four-fermion operators in [36]. Here I discuss the specific example of Q_1. It is to be understood, throughout the rest of this section, that the chiral limit is to be taken in the end.

From (19), in the large momentum limit, one may derive the renormalization conditions used in the NPM to fix the matrix Δ. For large p, the last term on the r.h.s. of eq. (19) is power supressed. This happens because the explicit m factor implies that the integrand has one less dimension that the other terms so that, at large momenta,

[2] The projectors obey the following orthogonality conditions: Tr $\mathrm{P}_i^{\pm}\Lambda_k^{\pm(0)} = \delta_{ik}$ and Tr $\mathrm{P}_i^{\pm}\mathcal{\Lambda}_k^{\pm(0)} = \delta_{ik}$ with $i,k = 1,\ldots,5$ where $\Lambda_k^{(0)\pm}$ and $\mathcal{\Lambda}_k^{(0)\pm}$ are the tree-level amputated Green functions of $Q_k^{\pm}$ and $\mathcal{Q}_k^{\pm}$ respectively.

it vanishes faster by one power of p. Moreover, in this limit the inverse quark propagator has the general form $S^{-1}(p) = i\Sigma_1 \gamma_\mu p_\mu$ (with Σ_1 a scalar form factor). This means that in this limit it anticommutes with γ_5. By projecting both sides of eq. (19) on the single operators Q_1, $\mathbf{P}_k$ (with $k = 1, \ldots, 5$) [34], one obtains

$$\left[\mathrm{Tr}\ \mathbf{P}_j \Lambda_1 + \sum_{k=1}^{5} \Delta_{1k} \mathrm{Tr}\ \mathbf{P}_j \Lambda_k \right] = \frac{\mathcal{Z}_{11}}{Z_{11}} \mathrm{Tr}\ \mathbf{P}_j \mathcal{J}\!\!\!\backslash_1 \quad (20)$$

which is precisely the system of equations used in [27] and [34] to fix the coefficients of the mixing matrix Δ.

The generalization to the other operators is straighforward, although more complicated. For example, considering the operators $Q_{2,3}$ and $\mathcal{Q}_{2,3}$, we have

$$[\Lambda(p) + \Lambda_\Delta(p) z_\Delta^T] z_\chi^T = -\frac{1}{2} [\mathcal{J}\!\!\!\backslash^{(1)} - \mathcal{J}\!\!\!\backslash^{(3)}] \zeta_\chi^T + \int d^4x \langle \mathcal{G}(p) \zeta_\chi^T 2\hat{m}\hat{P}(x) \rangle \prod_1^4 \langle S^{-1}(p) \rangle \quad (21)$$

where I have used a compact notation for the row-vectors $\mathcal{G} \equiv (\mathcal{G}_2, \mathcal{G}_3)$, $\Lambda \equiv (\Lambda_2, \Lambda_3)$, $\mathcal{J}\!\!\!\backslash \equiv (\mathcal{J}\!\!\!\backslash_2, \mathcal{J}\!\!\!\backslash_3)$, $\Lambda_\Delta \equiv (\Lambda_1, \Lambda_4, \Lambda_5)$ and for the mixing matrices

$$\begin{aligned} \zeta_\chi &= \begin{pmatrix} \mathcal{Z}_{22} & \mathcal{Z}_{23} \\ \mathcal{Z}_{32} & \mathcal{Z}_{33} \end{pmatrix} \\ z_\chi &= \begin{pmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{pmatrix} \\ z_\Delta &= \begin{pmatrix} \Delta_{21} & \Delta_{24} & \Delta_{25} \\ \Delta_{31} & \Delta_{34} & \Delta_{35} \end{pmatrix} \end{aligned} \quad (22)$$

Equation (21) correspond to a system of linear equations in the matrix elements of z_Δ and in $z_\chi \zeta_\chi^{-1}$ which can be used to fix the subtracted operators $\tilde{Q}_{2,3}$. Finally, overall renormalization conditions have to be imposed to the $\tilde{Q}_{2,3}$ to obtain the χRS mixing matrix z_χ.

Finally, I want to mention another method, implicitly suggested in [4,30], which has not been implemented yet in numerical simulations. This method is based on the Ward identities which can be written for gauge-invariant correlation functions. Let us introduce the "meson fields" $P^K(t_K) = \int d^3x \bar{s}(\vec{x}, t_k) \gamma_5 d(\vec{x}, t_k)$, $P^{\pi+}(t_\pi) = \int d^3x \bar{u}(\vec{x}, t_\pi) \gamma_5 d(\vec{x}, t_\pi)$, etc. By defining the following correlation functions

$$\begin{aligned} G_2(t_{\pi_1}, t_{\pi_2}) &= \langle 0 | P^{\pi_1}(t_{\pi_1}) \tilde{Q}(x_0) P^{\pi_2}(t_{\pi_2}) | 0 \rangle \\ \mathcal{G}_3(t_{\pi_1}, t_{\pi_2}) &= \int dt_K \langle 0 | P^K(t_K) \tilde{\mathcal{Q}}(x_0) \\ & P^{\pi_1}(t_{\pi_1}) P^{\pi_2}(t_{\pi_2}) | 0 \rangle \end{aligned} \quad (23)$$

the Ward identity can schematically be written as

$$\lim_{m \to 0} 2m \mathcal{Z} \mathcal{G}_3(t_{\pi_1}, t_{\pi_2}) = const. \times Z G_2(t_{\pi_1}, t_{\pi_2}) \quad (24)$$

where *const.* is a suitable constant which depends on the flavor quantum numbers of the operators Q and $\mathcal{Q}$ and of the external "meson fields". For different values of t_{π_1} and t_{π_2}, the above Ward identity results in a system of linear non-homogenous equations from which it is possible to extract $\Delta^\pm$ and $Z_\chi \mathcal{Z}_\chi^{-1}$. A study in this direction is underway [37].

4. OPE without OPE

In this section, I will discuss some new methods which have been developed in order to overcome the difficulties related to the renormalization of lattice composite operators relevant in many important physical applications. Among the others, let me mention the structure functions in deep inelastic scattering, kaon and B-meson non-leptonic exclusive decays, electromagnetic form factors at large momentum transfer, exclusive and inclusive semileptonic B-decays etc. The general idea underlying these new methods is essentially the same and I will denote it as Operator Product Expansion without Operator Product Expansion (OPE without OPE or OPEwOPE). The reason for choosing this name will be clearer in the following. I now discuss the general features of the OPEwOPE, starting from the description of the standard approach followed so far in lattice calculations.

Let us consider the short-distance OPE for the T-product of two generic currents between the physical external states $|A\rangle$ and $|B\rangle$

$$\langle B | T(J(x) J(0) | A \rangle \equiv \sum_i C_i(x, \mu) \langle B | \hat{O}_i(\mu) | A \rangle \quad (25)$$

where, for simplicity, Lorentz indices have been omitted. In the above equation, μ is the renormalization scale of the local, composite operators $\hat{O}_i(\mu)$ and the $C_i(x,\mu)$ are the Wilson coefficients, which can be computed in perturbation theory, provided $\mu \gg \Lambda_{QCD}$. The Wilson coefficients are choosen in such a way that the T-product is μ independent. The standard approach consists in computing the Wilson coefficients in perturbation theory and extracting the matrix elements of the local operators from suitable hadronic correlation functions. Many examples of matrix elements of local operators that have been studied in the past can be given: i) Moments of the structure functions, e.g. $\langle p|\bar{\psi}_i\gamma_{\mu_1} i\overleftrightarrow{D}_{\mu_2}\ldots i\overleftrightarrow{D}_{\mu_n}\psi|p\rangle$; ii) Hadronic weak decays ($K\to\pi\pi$ or $B\to\pi\pi$), e.g. $\langle\pi\pi|\left(\bar{s}\gamma_\mu^L d\right)\left(\bar{u}\gamma_L^\mu e\right)|K\rangle$; iii) K^0–$\bar{K}^0$ and B^0–$\bar{B}^0$ mixing, e.g. $\langle\bar{K}^0|\left(\bar{s}\gamma_\mu^L d\right)\left(\bar{s}\gamma_L^\mu d\right)|K^0\rangle$; iv) operators of the HQET entering inclusive and exclusive decay rates, e.g. $\langle B|\bar{h}_v D_0 h_v|B\rangle$ and $\langle B|\bar{h}_v(i\overrightarrow{D})^2 h_v|B\rangle$; iv) light-cone wave functions for electromagnetic form factors and exclusive $B\to\pi$ and $B\to\rho$ semileptonic decays, e.g. $\langle\pi|\bar{\psi}_i\gamma_{\mu_1}\gamma_5 i\overleftrightarrow{D}_{\mu_2}\ldots i\overleftrightarrow{D}_{\mu_n}\psi|0\rangle$.

The renormalized operators $\hat{O}_i(\mu)$ are obtained from the bare lattice operators either by using (boosted-tadpole improved) perturbation theory or non-perturbatively using the WIM or the NPM. The difficulties encountered in this approach are due to the fact that the ultraviolet behaviour of the "effective theory" (OPE, HQET, NRQCD, etc.) is much worse and divergences which were not present in the "full" theory appear in the different expansions. For example, if the currents are the SM quark currents responsible for weak decays, the l.h.s. of eq. (25) is finite, whereas the operators appearing on the r.h.s. are logarithmicaly or power divergent in the inverse lattice spacing a^{-1}. We briefly discuss examples taken from the cases considered above.

i) Given the bare energy-momentum tensor density $T^{\mu\nu}=\bar{\psi}(\gamma_\mu\overleftrightarrow{D}_\nu+\gamma_\nu\overleftrightarrow{D}_\mu)\psi$, the corresponding renormalized operator is given by

$$\hat{T}^{\mu\nu}(\mu)=Z(\mu)\left[T^{\mu\nu}+\frac{C(g_0^2)}{a}\delta_{\mu\nu}\bar{\psi}\psi\right] \qquad (26)$$

In this case it is easy to get rid of the power divergences by taking $\mu\neq\nu$.

ii) In presence of a charm-up GIM mechanism, the renormalized $O^\pm$ operators have the form [4,29]

$$\begin{aligned}\hat{O}^\pm(\mu)=Z^\pm(\mu)\Bigg[&O^\pm+\sum_i C_i^\pm(g_0^2)O_i\\&+(m_c-m_u)C_\sigma^\pm(g_0^2)\bar{s}\sigma^{\mu\nu}G^a_{\mu\nu}t^a d\\&+(m_c-m_u)(m_s-m_d)\frac{C_P^\pm(g_0^2)}{a}\bar{s}\gamma_5 d\\&+(m_c-m_u)\frac{C_S^\pm(g_0^2)}{a^2}\bar{s}d\Bigg]\end{aligned} \qquad (27)$$

where the O_i are operators of dimension six, the mixing of which is induced by the explicit chiral symmetry breaking of the lattice action and have been discussed in sec. 3. In this case, it has been impossible so far to get the matrix elements of the renormalized operators with resonable accuracy.

iii) By defining $S_0=\bar{h}_v D_0 h_v$ and $K=\bar{h}_v(i\overrightarrow{D})^2 h_v$, the renormalized operators have the form

$$\begin{aligned}\hat{S}_0(\mu)&=Z_h(\mu)\Big(\bar{h}_v D_0 h_v+\frac{C_h(g_0^2)}{a}\bar{h}_v h_v\Big)\\\hat{K}(\mu)&=Z_{D^2}(\mu)\Big(\bar{h}_v(i\overleftarrow{D})^2 h_v\\&+\frac{C_1(g_0^2)}{a}\bar{h}_v D_0 h_v+\frac{C_2(g_0^2)}{a^2}\bar{h}_v h_v\Big)\end{aligned} \qquad (28)$$

The subtraction procedure necessary to obtain a finite $\hat{S}_0$, which is related to the definition of the renormalized quark mass, is straightforward [38]. An accurate determination of the quark mass, however, requires the control of high orders in perturbation theory [39]; the definition of a finite $\hat{K}$, necessary to obtain the heavy quark kinetic energy is much harder and a small error on the final answer difficult to obtain [38].

The use of perturbation theory in the calculation of the mixing coefficients necessary to subract power divergencies gives unreliable results. A detailed discussion, with several examples can be found in [39]. For the calculation of logarithmicaly divergent or finite mixing coefficients instead, perturbation theory can be used, at least in principle. In practice, however, due to large higher-order corrections present in the expansion of the lattice perturbative series, the results are not very accurate and it is for this reason that

the non-perturbative methods discussed in sec. 3 have been developed.

All the problems connected to the construction of finite operators, out of the bare (power) divergent lattice ones, can be overcome, at least in principle, by using the OPEwOPE. I start by recalling a result presented at the Latt97 [30], and then discuss new proposals, and feasibility studies, which appeared this year.

Weak Hamiltonian The first example of OPEwOPE is related to non-leptonic weak decays. The standard construction of the weak Hamiltonian begins with the expression

$$\frac{g_W^2}{2}\int d^4x\, D^W_{\rho\nu}(x;M_{_W})T\left[J_{\rho L}(x)J^\dagger_{\nu L}(0)\right] \qquad (29)$$

where $D^W_{\rho\nu}(x;M_{_W})$ is the W-boson propagator and $J_{\rho L}$ is the (left-handed) hadronic weak current. One then performs the OPE of the product of the two currents in eq. (29)

$$\langle B|\mathcal{H}^W_{\rm eff}|A\rangle = \frac{G_F}{\sqrt{2}}\sum_i C_i(\mu,M_{_W})M_{_W}^{6-d_i}\langle B|\hat{O}^{(i)}(\mu)|A\rangle \qquad (30)$$

where d_i is the dimension of the operator $\hat{O}^{(i)}(\mu)$, and the functions $C_i(\mu,M_{_W})$ result from the integration of the Wilson expansion coefficients, $c_i(x;\mu)$ (defined in eq. (32) below), with the W-propagator. Schematically, suppressing Lorentz indices, one has

$$\frac{C_i(\mu,M_{_W})}{M_{_W}^{d_i-6}} = \int d^4x\, D^W(x;M_{_W})c_i(x;\mu) \qquad (31)$$

The sum in the expansion (30) is over operators of increasing dimension. In the following only operators with dimensions $d_i \le 6$ are considered, since the contribution from operators with $d_i > 6$ is suppressed by powers of $1/M_{_W}$.

All the intricacies of operator mixing in the definition of the finite and renormalized operators, $\hat{O}^{(i)}(\mu)$, come about because the integral in (29) is extended down to the region of extremely small x. The complicated mixing for the $\hat{O}^{(i)}(\mu)$'s in terms of bare operators arises from contact terms when the separation of the two currents goes to zero (i.e. when $|x|$ is of the order of a). This observation suggests that a simple way to avoid these complications is to define the renormalized operators by enforcing the OPE for distances $|x|$ much larger than the lattice spacing a. This is done by computing directly the T-product of the two currents rather than the matrix elements of the local operators on the r.h.s. of eq. (30). For this reason I call this method OPEwOPE. We imagine proceeding in the following way: 1) Take the T-product of two properly normalized weak currents, $J_{\rho L}(x)J^\dagger_{\rho L}(0)$. If required these currents can be improved. 2) Measure the hadronic matrix element $\langle B|T[J_{\rho L}(x)J^\dagger_{\rho L}(0)]|A\rangle$ in a Monte Carlo simulation, as a function of x for $|x|\to 0$ in the region $a \ll |x| \ll \Lambda^{-1}_{QCD}$. 3) Extract the numbers $\langle B|\hat{O}^{(i)}(\mu)|A\rangle$ by fitting in the region defined above the x-behaviour of $\langle B|T[J_{\rho L}(x)J^\dagger_{\rho L}(0)]|A\rangle$ to the formula

$$\langle B|T\left[J_{\rho L}(x)J^\dagger_{\rho L}(0)\right]|A\rangle = \sum_i c_i(x;\mu)\langle B|O^{(i)}(\mu)|A\rangle\,, \qquad (32)$$

where the Wilson coefficients $c_i(x;\mu)$ are determined by continuum perturbation theory using any standard renormalization scheme. Since we only consider operators of dimension 6 or lower, the T-product differs from the right-hand side of eq. (32) by terms of $O(|x|^2\Lambda^2_{QCD})$, which is an estimate of the size of the systematic errors in this procedure. 4) Insert the numbers $\langle B|\hat{O}^{(i)}(\mu)|A\rangle$ determined in this way into the expression for the matrix elements of $\mathcal{H}^W_{\rm eff}$ in eq. (30).

For the implementation of this procedure, what is required is the existence of a window, in which the distance between the two currents is small enough, so that perturbation theory can be used to determine the expected form of the OPE, but large enough that lattice artifacts are small. The existence of a window is a necessary conditions in all applications of the OPEwOPE.

The method determines directly the "physical" matrix elements of the operators appearing in the OPE of the two currents, i.e. the matrix elements of the finite, renormalized operators $\hat{O}^{(i)}(\mu)$, without any reference to the magnitude of the W-mass. Thus we do not need to probe distances of $O(1/M_{_W})$ with lattice calculations. The μ-dependence of the matrix elements of the oper-

ators $\hat{O}^{(i)}(\mu)$ is given trivially by that of the (perturbative) Wilson coefficients, $c_i(x;\mu)$. It compensates the related μ-dependence of the functions $C_i(\mu, M_{_W})$ in such a way that the l.h.s of eq. (30) is independent of the choice of the subtraction point. A similar comment holds for the dependence on renormalization scheme.

Non-linear σ-model A feasibility study of this method, in a simplified case, has be presented by S. Caracciolo in a parallel session at this Conference [40]. They have considered the two-dimensional non-linear σ-model, for which it is possible to define the following conserved currents

$$j_\mu^{a,b}(x) = \frac{1}{g}\left(\sigma^a(x)\partial_\mu\sigma^b(x) - \sigma^b(x)\partial_\mu\sigma^b(x)\right) \quad (33)$$

where $\vec{\sigma} \equiv (\sigma^1, \ldots, \sigma^N)$ with the condition $\vec{\sigma}(x)\cdot\vec{\sigma}(x) = 1$. The relevant expression is

$$\langle p|\sum_{a,b} j_\mu^{a,b}(x) j_\nu^{a,b}(0)|p\rangle = C_T(x;\mu)\langle p|\hat{T}_{\mu\nu}|p\rangle \quad (34)$$

where $|p\rangle$ is some external state of momentum p and $\hat{T}_{\mu\nu}$ the renormalized energy-momentum tensor ($T_{\mu\nu} = \partial_\mu\vec{\sigma}\cdot\partial_\nu\vec{\sigma}$). This operator has zero anomalous dimension, but it is subject to a finite renormalization (analogous to $Z_{V,A}$ in the case of Wilson fermions) on the lattice.

In the OPEwOPE approach, one computes non-perturbatively the matrix elements of the T-product of the original currents, corresponding to the l.h.s of (34) and the Wilson coefficients (in this case there is only one operator corresponding to $C_T(x;\mu)$) in perturbation theory. From the knowledge of these quantities, one can then extract the desired matrix element of the local operators, i.e. $\langle p|\hat{T}_{\mu\nu}|p\rangle$. In the example considered here, we may also compute directly the matrix element of $\hat{T}_{\mu\nu}$ and check to what extent the procedure works. In [40] they found a large window of distances for which the OPE is verified, when continuum perturbation theory is used in the calculation of the Wilson coefficient, giving the correct matrix element independently of the external states.

The quark propagator The simplest T-product that can be considered is the quark propagator in a fixed gauge, for example the Landau gauge. Let us consider for example the following quantity

$$\Sigma_2(x) = \langle 0|\psi(x)^\alpha_A\bar{\psi}(0)^\alpha_A|0\rangle\,, \quad (35)$$

where α and A are spin and color indices respectively. At short distances, i.e. as $|x| \to 0$, we have

$$\Sigma_2(x) \to C_\psi(x;\mu)\langle\bar{\psi}\psi(\mu)\rangle + C_m(x;\mu)\frac{\hat{m}(\mu)}{|x|^2} + \mathcal{O}\left(\frac{a}{|x|^4}\right) \quad (36)$$

where $C_\psi(x;\mu)$ and $C_m(x;\mu)$ are the Wilson coefficients, $\langle\bar{\psi}\psi(\mu)\rangle$ the quark condensate and $\mathcal{O}(a/|x|^4)$ represent (gauge non-invariant) lattice artifacts [41]. The Wilson coefficients, which are gauge-dependent, can be computed in any continuum renormalization scheme, for example in the $\overline{MS}$ scheme [42]. In the chiral limit, by fitting $\Sigma_2(x)$ to (36), and using $C_\psi(x;\mu)$ and $C_m(x;\mu)$ from perturbation theory, one can extract the value of the condensate and the renormalized quark mass in the same scheme as the Wilson coefficients. It would be very interesting to compare the values obtained in this way, with those obtained with other techniques. Note that if we use the Fourier transform of Σ_2, there are further contributions from contact terms, which can make the extraction of the condensate and of the quark mass more complicated.

The shape function The knowledge of the shape function $f(k_+)$ is a fundamental ingredient for the extraction of $|V_{ub}|$ from the end-point of the lepton spectrum. The same function also enters the calculation of the photon spectrum in radiative B decays. To give an explicit example, the differential distribution in semileptonic decays is given by

$$\frac{d\Gamma}{dE_\ell} \equiv \int_0^{M_B} dk_+\; f(k_+)\;\frac{d\Gamma_{PM}}{dE_\ell}(m_b^*, E_\ell)$$
$$= |V_{ub}|^2\frac{G_F^2}{12\pi^3}\;E_\ell^2\int_0^{M_B} dk_+\; f(k_+)$$
$$\Theta(m_b^* - 2E_\ell)\left[3{m_b^*}^2 - 4m_b^*E_\ell\right]\,, \quad (37)$$

where $m_b^* = M_B - k_+$.

The physical rate can be derived from the imaginary part of the forward matrix element of the

T-product of two weak currents

$$W^{\mu\nu} = \frac{1}{\pi}\mathrm{Im} i \int d^4x\ e^{-iq\cdot x} \times \langle \bar{B}(v)|T\Big[J^{\mu\ \dagger}(x)\ J^{\nu}(0)\Big]|\bar{B}(v)\rangle \quad (38)$$

In the standard approach, one applies the OPE to the T-product above, and $W^{\mu\nu}$ is written in terms of the following matrix elements of local operators

$$\mathcal{M}_n \propto \langle \bar{B}(v)|\bar{b}_v\gamma^{\nu}(iD^{\mu_1})\dots(iD^{\mu_n})b_v|\bar{B}(v)\rangle \quad (39)$$

which correspond to the moments of the shape function

$$\mathcal{M}_n = \int_0^{M_B} dk_+ k_+^n f(k_+) \quad (40)$$

With the OPEwOPE, it is possible to obtain the full shape function. The final result of [43] is very simple. One study the ratio of two correlation functions

$$W^{\mu\nu}(t,\vec{Q}) = \lim_{t_f,t_i\to\infty} \frac{W^{\mu\nu}_{t_f,t_i}(t,\vec{Q})}{S_{t_f,t_i}} e^{-M_B t}, \quad (41)$$

where the numerator corresponds to insertion of the three-dimensional Fourier transform of the (Euclidean) T-product for $t > 0$

$$W^{\mu\nu}_{t_f,t_i}(t,\vec{Q}) = \frac{1}{\pi}\int d^3x\ e^{-i\vec{Q}\cdot\vec{x}} \times \langle 0|\Phi^{\dagger}_{\vec{p}_B=0}(t_f)J^{\dagger}_{\mu}(\vec{x},t)J_{\nu}(0)\Phi_{\vec{p}_B=0}(-t_i)|0\rangle$$

and the denominator is the usual B-meson propagator

$$S_{t_f,t_i} = \langle 0|\Phi^{\dagger}_{\vec{p}_B=0}(t_f)\Phi_{\vec{p}_B=0}(-t_i)|0\rangle \quad (42)$$

$\Phi_{\vec{p}_B}(t)$ is the B interpolating field with definite spatial momentum $\vec{p}_B$

$$\Phi_{\vec{p}_B}(t) = \int d^3x e^{-i\vec{p}_B\cdot\vec{x}}\Phi_B(\vec{x},t) \quad (43)$$

In terms of $f(k_+)$, we have

$$W^{\mu\nu}(t,\vec{Q}) = \frac{1}{2\pi}\int_0^{M_B} dk_+\ f(k_+)\frac{e^{-\left(k_+ + \sqrt{\vec{Q}^2}\right)t}}{2\sqrt{\vec{Q}^2}} \times \left(\bar{Q}^{\mu}\delta^{\nu 0} + \bar{Q}^{\nu}\delta^{\mu 0} - g^{\mu\nu}\bar{Q}^0 - i\epsilon^{0\mu\nu\alpha}\bar{Q}_{\alpha}\right) \quad (44)$$

where $Q_0^+ = k_+ + \sqrt{k_+^2 + \vec{Q}^2} \sim k_+ + \sqrt{\vec{Q}^2}$. By a suitable choice of the Lorentz components μ and ν of the currents and of the spatial momentum $\vec{Q}$, and by studying the time dependence of $W^{\mu\nu}(t,\vec{Q})$, we can unfold the integral above and extract the shape function. As explained in [43], the same method can be used to extract the structure functions of deep inelastic scattering.

Light-cone wave-funtions The light-cone wave functions allow to predict form factors relevant in many exclusive processes, such as electromagnetic elastic scattering at large momentum transfer, or exclusive semi-leptonic decays as $B \to \pi$ ($B \to \rho$) and $B \to K^*\gamma$ decays [44]–[48]. The knowledge of the relevant light-cone functions would allow the determination of the form factors at $q^2 \sim 0$, a kinematical region which is not accessible with standard lattice techniques. In [49], it was shown that they can be computed, analogously to the shape function, by suitable ratios of three- and two-point correlation functions of Euclidean T-product. In this case the three-point function corresponds to the insertion of the three-dimensional Fourier transform of the T-product

$$\langle \pi^-|T\Big[\bar{d}(x)\exp^{i\int_0^s A_{\mu}(s'n^{\mu})n^{\mu}ds'}\ \gamma_{\mu}\gamma_5 u(0)\Big]|0\rangle \quad (45)$$

where $n^{\mu} = (q^{\mu} + p_{\pi}^{\mu})/\sqrt{-q^2}$ and $x^{\mu} = sn^{\mu}$. The OPE of the T-product can be written in terms of matrix elements of local operators

$$\mathcal{M}_n \propto \langle \pi^-|\bar{d}(0)\gamma^{\mu}\gamma_5(iD^{\mu_1})\dots(iD^{\mu_n})u(0)|0\rangle \quad (46)$$

which, as for the shape function considered before, corresponds to the moments of the ligh-cone wave-fuction

$$\mathcal{M}_n = \int_0^1 du\ u^n\Phi_{\pi}(u) \quad (47)$$

The matrix elements of the local operators corresponding to the first two non-trivial moments were studied, using the standard approach in [50]. With the OPEwOPE one can obtain the full light-cone wave function Φ without the problems associated with the renormalization of the local operators of eq. (46) [49].

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